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Existence of smooth utilities on Banach lattices

Manuel Besada, Javier García, Miguel Mirás*, Carmen Vázquez

Departamento de Matemáticas, Facultade de Economía, Universidade de Vigo, Lagoas de Marcosende (SN), 36200 Vigo, Pontevedra, Spain

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Abstract

In this paper, we focus our attention on the representability of a preference relation by differentiable utility functions when the consumption sets belong to an infinite dimensional commodity space. We obtain sufficient conditions for the existence of a C^r function representing a preference relation defined on an open subset of a Banach lattice. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The three main concepts in the theory of consumer behavior: preference relations, utility functions and demand functions, are closely related. Naturally, understanding the relationships among them is a fundamental goal of this theory. Many problems of consumer decision making are better treated if the consumer's preference relation is represented by a differentiable utility function, see, for instance, Bridges and Mehta (1995). When differentiability is added to the picture, knowing if the regularity of one of those concepts implies the regularity of the others becomes essential.

The problem, for a finite number of commodities, was solved in Debreu (1972, 1976), and Mas-Colell (1985), by giving necessary and sufficient conditions for the smoothness of the demand and the utility functions. But a wide variety of infinite dimensional spaces arise naturally in economics. Mas-Colell and Zame (1991) provide some illustrative examples. When infinitely many commodities are considered, results as those mentioned before for the finite case are scarce. Few things are known about the existence and smoothness of

* Corresponding author. Tel.: +34-986812449; fax: +34-986812401.
E-mail address: mmiras@uvigo.es (M. Mirás).

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demand functions with an infinite number of commodities. One of the known facts, due to Araujo (1988), states that if the commodity space cannot be given an equivalent norm that comes from an inner product, then one cannot find a single C^2, strictly quasi-concave utility function that gives rise to a continuously differentiable demand function defined on its dual. A fundamental assumption is made throughout the cited article: the existence of a smooth utility.

Our aim in this paper is to impose conditions on the preference relation that can assure the existence of a smooth utility function. Properties of this kind could be very useful in deriving new and economically interesting results on consumer behavior in large commodity spaces.

The mathematical model of an abstract commodity space is a Banach lattice, see Araujo (1988), Richard and Zame (1986) or Mas-Colell (1975), among others. Now, allow us a brief comment about the difficulties of generalizing properties, which are based on differentiability, from finite dimensional economies to infinite dimensional ones. All differentiability concepts require that the consumption set must be open. Typically, the preference relation is defined on a subset of the positive cone of the commodity space. But, generally speaking, in infinite dimensional spaces, the positive cone has (ℓ_∞ is an exception) empty interior.

Trying to solve this problem and, at the same time, keep the scope of our analysis as wide as possible to include the most general cases, we consider two alternative kind of spaces depending on whether the positive cone has empty interior or not. In the first case, the preference relation will be defined on an open neighborhood of the positive cone. Naturally, in the second case, the consumption set is the interior of the positive cone. This approach has been adopted, for instance in Araujo (1988) and Besada and Vázquez (1999). Using the same arguments, Richard and Zame (1986) developed an extension of uniformly proper preferences to a certain larger set with non-empty interior.

In this setting, we prove in Theorem 1, under mildly assumptions, that any C^r preference relation on an open consumption set of a Banach lattice that has non-empty intersection with the positive cone, can be represented by a C^r utility function. Furthermore, Theorem 2 states that this utility representation is unique up to C^r diffeomorphisms.

2. Assumptions and notations

The commodity space is a Banach lattice E and E_+ represents its positive cone. Given a linear functional $L : E \to \mathbb{R}$, the relation $L > 0$ means that $L(x) \geq 0$ for all $x \in E_+$ and there is $z \in E_+$ such that $L(z) > 0$.

Let $\succeq$ denote a complete, transitive and reflexive binary relation defined on a subset X of E. Obviously, $\succeq$ is the preference relation defined on the consumption set X. The membership $(x, z) \in \succeq$ is usually written as $x \succeq z$, and it is read: "the bundle x is at least as good as the bundle z". When $x \succeq z$ and $z \succeq x$ both hold at the same time, we write $x \sim z$ and say that "x is indifferent to z". We put $x \succ z$, whenever $x \succeq z$ and x is not indifferent to z.

Given any $x \in X$, the set of all the bundles which are indifferent to x, $I_x = \{z \in X : x \sim z\} \subset E$, is called the indifference set of x. We denote by $I = \{(x, z) \in X \times X : x \sim z\} \subset E \times E$.

Our results rely on an archimedean-like property of the preference relation. We say that $\succeq$ is *non-discriminatory* if for each $x, z \in X$, there exist $r, s \in \mathbb{R}$ such that $rs > 0$ and $rz \succ x \succ sz$.

The preference relation $\succeq$ has a utility representation if there exists a function $u : X \to \mathbb{R}$, called a utility function, such that for all $x, z \in X$, $x \succeq z$ if and only if $u(x) \geq u(z)$.

The preference relation $\succeq$ is *continuous* if the sets $\{x \in X : x \succeq z\}$ and $\{x \in X : z \succeq x\}$ are closed in X for all $z \in X$; and is *monotone* whenever $x > z$ implies $x \succ z$.

3. Results

Let us start by borrowing from Debreu (1972) the concept of a smooth preference relation when the commodity space is a Banach lattice.

Definition 1. A preference relation $\succeq$ defined on an open subset X of a Banach lattice E is said to be C^r if the set I is a C^r submanifold of $E \times E$.

Our main result, Theorem 1, states a sufficient condition for the existence of a smooth utility function, without critical points, representing a C^r preference relation. The proof is constructive and quite long; basically, a reiterated application of the implicit function theorem. We divide the proof in several steps. First, we see that the result holds on a neighborhood U of each commodity bundle $x_0 \in X$. After that, any two local utilities for two indifferent bundles x_1 and x_2, defined on neighborhoods with non-empty intersection, are transformed into a new local utility defined on an open set containing x_1 and x_2. Next, we show that each $x \in X$ is indifferent to a unique bundle αe that lies in the ray of a fixed bundle $e \in X$. Another local utility is defined on an open set containing both x and αe. Finally, we find the smooth global utility function we were looking for.

Theorem 1. *Let $\succeq$ be a monotone, continuous, non-discriminatory and C^r preference relation defined on an open subset X of a Banach lattice E. If $X \cap E_+ \neq \emptyset$ and all the indifference sets of $\succeq$ are path-connected, there exists a C^r utility function $u : X \to \mathbb{R}$ such that $Du(x) > 0$ for all $x \in X$.*

Proof. We begin by constructing, in a simple way, the utility function u. Consider a fixed bundle $e \in X \cap E_+$. First of all, we see that each indifference set intersects the ray $\{te : t \in \mathbb{R}\}$ in just one point.

Step 1: For each $x \in X$ there exists a unique real number $u(x) \in \mathbb{R}$ such that $u(x)e \sim x$.

For each $x \in X$, consider the set $S = \{t \neq 0 : te \succ x\}$. According to the non-discriminatory property, there are real numbers $r, s \in \mathbb{R}$, $r, s > 0$, such that $re \succ x \succ se$. Therefore, $r \in S$ and $S \neq \emptyset$. In addition, s is a lower bound of S because if there exists $t \in S$ such that $s > t$, then $se > te$, since $e \in E_+$, and by the monotonicity of the preference relation, $se \succ te \succ x$. Now, we can define $u(x) = \inf S$. Obviously, $u(x) \in \mathbb{R}$ because S is non-empty and bounded below.

Suppose that $u(x)e \succ x$. In this case, $u(x)e \in A = \{z \in X : z \succ x\}$. The preference relation is continuous, so the set A is open. Thus, there is $\varepsilon > 0$ such that $te \succ x$ whenever $t \in (u(x) - \varepsilon, u(x) + \varepsilon)$. Consequently, if $t \in (u(x) - \varepsilon, u(x))$, then $t \in S$ and $t < u(x)$, which is absurd because $u(x)$ is the infimum of S.

Conversely, suppose that $x \succ u(x)e$. Once again, using the continuity of $\succeq$, there is $\varepsilon > 0$ such that $x \succ te$ whenever $t \in (u(x) - \varepsilon, u(x) + \varepsilon)$. But, on the other hand, since $u(x) = \inf S$, $te \succeq x$ if $t \in (u(x), u(x) + \varepsilon)$. This leads to a contradiction.

In summary, we have stablished that $u(x)e \sim x$. Finally, the monotonicity of the preference relation implies immediately the uniqueness. This concludes step 1.

Clearly, the function $u : X \to \mathbb{R}$ is a utility function representing the preference relation $\succeq$. It just remains to be checked that this utility u is, in fact, a C^r function.

Step 2: For each $x_0 \in X$, there exist W_{x_0}, an open neighborhood of x_o, and a C^r function $v : U \to \mathbb{R}$ such that $Dv(x) > 0$ for all $x \in W_{x_0}$. In addition, v is a utility function representing the preference relation $\succeq$ restricted to W_{x_0}.

If $x_0 \in X$, then $(x_0, x_0) \in I$. Hence, by the differentiability assumption, there exist W, an open neighborhood of x_0 in E, a C^r diffeomorphism $g : W \times W \to V$, with V an open set in $E \times E$, and H, a hypersurface in $E \times E$, such that $g((W \times W) \cap I) = H \cap V$. As H is a hypersurface, H is the kernel of a continuous linear function $\varphi : E \times E \to \mathbb{R}$, that is, $H = \ker\varphi$. Therefore, if we define $h = \varphi \circ g$,

$$x \sim z \Leftrightarrow \varphi(g(x, z)) = 0 \Leftrightarrow h(x, z) = 0, \quad (x, z) \in W \times W. \tag{1}$$

Given $w \in E$ and $(x, z) \in W \times W$, we can write

$$\begin{aligned} D_{(w,w)}h(x, z) &= Dh(x, z)(w, w) = Dh(x, z)(w, 0) + Dh(x, z)(0, w) \\ &= D_{(w,0)}h(x, z) + D_{(0,w)}h(x, z). \end{aligned}$$

If, in addition, $x \sim z$, then we deduce directly from (1) that

$$D_{(w,w)}h(x, x) = \lim_{t\to 0} \frac{h(x + tw, x + tw) - h(x, x)}{t} = 0.$$

Combining the last two expressions, yields that, for all $w \in E$ and $x, z \in W$, $x \sim z$,

$$D_{(w,0)}h(x, z) = -D_{(0,w)}h(x, z). \tag{2}$$

In particular, the next equalities are valid

$$Dh(x_0, x_0) = D(\varphi \circ g)(x_0, x_0) = D\varphi(g(x_0, x_0)) \circ Dg(x_0, x_0) = \varphi \circ Dg(x_0, x_0).$$

Thus, $Dh(x_0, x_0)$ is onto, because φ is onto and $Dg(x_0, x_0)$ is an isomorphism. Consequently, $D_{(\bar{w},0)}h(x_0, x_0) \neq 0$ for some $\bar{w} \in E_+$. Let $E_{\bar{w}}$ be the topological supplementary of $\langle\bar{w}\rangle$ (the ray generated by $\bar{w}$) in E. Then, the vector x_0 admits the decomposition $x_0 = x_0^1\bar{w} + x_0^2$ and, furthermore, the domain of h can be viewed as $\langle\bar{w}\rangle \times E_{\bar{w}} \times E = \mathbb{R} \times E_{\bar{w}} \times E$. Now, consider the function $F : \mathbb{R} \times E \to \mathbb{R}$, $F(x^1, x) = h(x^1\bar{w} + x_0^2, x)$. We have that, $F \in C^r$, $F(x_0^1, x_0) = h(x_0, x_0) = 0$ and

$$DF(x^1, x) = \begin{matrix} D_1h(x^1\bar{\omega} + x_0^2, x)\bar{\omega} \\ D_2h(x^1\bar{\omega} + x_0^2, x) \end{matrix},$$

for all $(x^1, x) \in \mathbb{R} \times E$. Obviously, $\frac{\partial F}{\partial x^1}(x_0^1, x_0) = D_{(\bar{w},0)}h(x_0, x_0) \neq 0$. Therefore, the implicit function theorem guaranties the existence of a C^r function $v : W_{x_0} \to \mathbb{R}$, defined on a neighborhood $W_{x_0} \subset W$ of x_0, such that, for all $x \in W_{x_0}$, $F(v(x), x) = h(v(x)\bar{w} + x_0^2, x) = 0$ and

$$Dv(x) = -\frac{1}{D_1 h(v(x)\bar{w} + x_0^2, x)\bar{w}} D_2 h(v(x)\bar{w} + x_0^2, x).$$

Therefore, $v(x)\bar{w} + x_0^2 \sim x$ for all $x \in W_{x_0}$. This fact and the monotonicity property imply that $v : W_{x_0} \to \mathbb{R}$ is a utility function representing the preference relation $\succeq$ restricted to W_{x_0}.

The utility function v is increasing, so $Dv(x)(w) \geq 0$ for all $x \in W_{x_0}$ and $w \in E_+$. Furthermore, using relation (2), one can easily prove that $Dv(x)(\bar{w}) = 1$ for all $x \in W_{x_0}$. Consequently, $Dv(x) > 0$ for all $x \in W_{x_0}$.

Step 3: Given $x_1, x_2 \in X$, $x_1 \sim x_2$, let $v_1 : W_1 = W_{x_1} \to \mathbb{R}$, $v_2 : W_2 = W_{x_2} \to \mathbb{R}$ be two C^r local utility functions, constructed as in step 2, with $W_1 \cap W_2 \neq \emptyset$. If the indifference manifold I_{x_1} contains the trace of a curve $I(x_1, x_2)$ on $W_1 \cup W_2$ beginning at x_1 and ending at x_2, then there exists a C^r local utility function, v, defined on an open set A contained in $W_1 \cup W_2$ and containing $I(x_1, x_2)$, such that $Dv(x) > 0$ for all $x \in A$.

Choose $z \in I(x_1, x_2) \cap (W_1 \cup W_2)$. Note, that this intersection is non-empty because $I(x_1, x_2)$ is connected. The set $W_1 \cup W_2$ is open. Therefore, there exists $t_0 > 0$ such that $\{z + t\bar{w}_1 : t \in (-t_0, t_0)\} \subset W_1 \cap W_2$, being $\bar{w}_1$ the unitary vector used to define the local utility v_1 in step 2. Consider the function $\phi : (-t_0, t_0) \to \mathbb{R}$, $\phi(s) = v_1(z + s\bar{w}_1)$ and the open set $B = \{x \in X : z + t_0\bar{w}_1 \succ x \succ z - t_0\bar{w}_1\}$. Straightforward calculations show that $\phi'(s) = Dv_1(z + s\bar{w}_1)(\bar{w}_1) = D_{\bar{w}_1} v_1(z + s\bar{w}_1) = 1$, for all $s \in (-t_0, t_0)$. Therefore, ϕ is a C^r diffeomorphism and $\phi(s) \in (\phi(-t_0), \phi(t_0))$ for all $s \in (-t_0, t_0)$. In particular, $v_1(z) \in (\phi(-t_0), \phi(t_0))$, or equivalently, $z \in B$. Also, the trace of the curve $I(x_1, x_2)$ is contained in B.

On the other hand, if $x \in W_1 \cap B$, $v_1(x) \in (v_1(z - t_0\bar{w}_1), v_1(z + t_0\bar{w}_1))$. Thus, $\phi^{-1}(v(x)) \in (-t_0, t_0)$ and $z + \phi^{-1}(v(x))\bar{w}_1 \in W_1 \cap W_2$. According to all the previous considerations, the function $v : A = (W_1 \cup W_2) \cap B \to \mathbb{R}$ given by

$$v(x) = \begin{cases} v_2(x), & \text{if } x \in W_2 \cap B \\ v_2(z + \phi^{-1}(v_1(x))\bar{w}_1), & \text{if } x \in W_1 \cap B \end{cases}$$

is the desired local utility function.

Step 4: For every $x_0 \in X$ there exists a C^r local utility function $v_{x_0} : U_{x_0} \to \mathbb{R}$ defined on an open set U_{x_0} containing both x_0 and $u(x_0)e$ such that $Dv_{x_0}(x) > 0$ for all $x \in U_{x_0}$.

Fix $x_0 \in X$ and put $\alpha = u(x_0)$. The indifference manifold I_{x_0} is path-connected, so let $I(x_0, \alpha e)$ be the trace of a curve on I_{x_0} beginning at x_0 and ending at αe. According to step 2, for every point $z \in I(x_0, \alpha e)$, there are a neighborhood of z, W_z, and a utility function $v_z : W_z \to \mathbb{R}$ representing the preference relation restricted to W_z. The set $I(x_0, \alpha e)$ is compact so that $I(x_0, \alpha e) \subset \bigcup_{i=1}^{k} U_{x_i}$ for some finite number of points $\{x_1 = x_0, x_2, \ldots, x_k = \alpha e\} \in I(x_0, \alpha e)$. A reiterated application of the previous step produces a utility function with all the wanted properties.

Step 5: The utility function u is differentiable at each $x_0 \in X$.

For each $x_0 \in X$, let α, U_{x_0} and the utility function v be those given by step 4. Define the function $G : U_{x_0} \times U(\alpha) \to \mathbb{R}$, $G(x, t) = v(x) - v(te)$, where $U(\alpha)$ is any open neighborhood of α such that $te \in U_{x_0}$ for all $t \in U(\alpha)$. Certainly, $G \in C^r$, $G(x_0, \alpha) = 0$ and $DG(x, t) = (Dv(x) \quad - Dv(te)e)$ for all $t \in I(\alpha)$. But, $\frac{\partial F}{\partial t}(x_0, \alpha) = -Dv(\alpha e)(e) \neq 0$, thus, the implicit function theorem yields an open neighborhood of x_0, $\tilde{U}_{x_0}$, and a C^r function, $\tilde{u}_{x_0} : \tilde{U}_{x_0} \to \mathbb{R}$, such that $F(x, \tilde{u}_{x_0}(x)) = 0$ for all $x \in \tilde{U}_{x_0}$, i.e. $x \sim \tilde{u}_{x_0}(x)e$. One can easily argue that u coincides with $\tilde{u}_{x_0}$ in a neighborhood of x_0. Indeed, as we have remarked at the beginning of the proof, $u(x)$ is the only real number such that $x \sim u(x)e$. Therefore, u is C^r in X. □

Some brief comments on the hypothesis of Theorem 1 are in order. Observe that the concept of a non discriminatory preference relation is not equivalent to the classical archimedean property. Nevertheless, many classes of preference relations are non-discriminatory. For instance, every monotone relation defined on the interior of the positive cone of either $\mathbb{R}^n$ or ℓ_∞ is non-discriminatory. In general, if E is a Banach lattice with a positive Schauder base $\{e_k : k \in \mathbb{N}\}$ then every monotone and continuous preference relation on $E_{++} = \{\sum_{k=1}^{\infty} x_k e_k : x_k > 0, \text{for all } k \in \mathbb{N}\}$ is non-discriminatory. Clearly, a non-discriminatory preference relation is non-satiated. On the other hand, the technical condition on the connectedness of the indifferent sets is met if, for instance, the preference relation is strictly convex.

The previous theorem stablishes the existence of a smooth utility function representing the preference relation, but such a function may not be unique. The next result reveals how two different smooth utility functions representing the same preference relation are related.

Theorem 2. *Let u and v be two C^r utility functions representing a preference relation $\succeq$ defined on an open subset X of a Banach lattice E. If $Du(x) > 0$ and $Dv(x) > 0$ for all $x \in X$, there exist a C^1, strictly increasing function $\varphi : v(X) \to \mathbb{R}$, such that $u = \varphi \circ v$. In addition, $\varphi'(t) \neq 0$ for all $t \in v(X)$.*

Proof. If $t \in v(X)$ then $t = v(x)$ for some $x \in X$, so we can define the value $\varphi(t) = u(x)$. The function $\varphi : v(X) \to \mathbb{R}$ is a well defined and strictly increasing function such that $u = \varphi \circ v$. We claim that φ is C^1 and that $\varphi'(t) \neq 0$ for all $t \in V(X)$.

Let $t_0 \in v(X)$ and $x_0 \in X$ be such that $t_0 = v(x_0)$. We know that $Du(x_0) > 0$ and $Dv(x_0) > 0$, then $Du(x_0)(w_0)Dv(x_0)(w_0) \neq 0$ for some vector $w_0 \in E_+$. Write $q = Dv(x_0)(w_0) \in \mathbb{R}$ and define the function

$$F : U \times \mathbb{R} \to \mathbb{R}, \quad F(\lambda, t) = v(x_0 + \lambda q w_0) - t,$$

where U is any suitable neighborhood of the origin. One can easily check that $D_\lambda F(0, t_0) = qDv(x_0)(w_0) = q^2 \neq 0$. Then the implicit function theorem implies the existence of a neighborhood V_0 of t_0 and a C^1 function, $\lambda : V_0 \to \mathbb{R}$, verifying that $\lambda(t_0) = 0$, $v(x_0 + \lambda(t)qw_0) - t = 0$ for all $s \in U$ and $\lambda'(t_0) = \frac{1}{q^2}$. Finally, consider the function $\mu : U \to \mathbb{R}$, $\mu(t) = u(x_0 + \lambda(t)qw_0)$. It follows immediately that μ agrees with φ on U and $\mu'(t_0) = \frac{Du(x_0)w_0}{q} \neq 0$. □

Therefore, if $\succeq$ is a C^r preference relation defined on an open subset of a Banach lattice and u, v are two C^r utility functions without critical points representing $\succeq$, then there exists a C^r diffeomorphism φ such that $u = \varphi \circ v$.

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