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# The generalized marginal rate of substitution

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**Abstract**

The generalized marginal rate of substitution concept is defined and related to the uniform properness property of a preference relation. 

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## 1. Introduction

In the modern theory of consumer behaviour, the marginal rate of substitution (MRS) is employed to measure the relative marginal utility. For a preference relation defined on a subset $X$ of $\mathbb{R}^n$, this MRS between two commodities $i$ and $j$, at a point $x=(x_1,\dots,x_n)$, is defined as the quantity of commodity $j$ that would just compensate the consumer for the loss of a marginal unit of commodity $i$. This is true provided that the quantity of all other commodities and the consumer's level of satisfaction are held constant. If the utility function $u$ is smooth this definition of $\mathrm{MRS}_{ij}(x)$ turns out to be mathematically equivalent to the ratio $(u_i)/(u_j)$ where $u_k$ is the partial derivative of $u$ with respect to commodity $k$. It is

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important to note that MRS is defined subject to the conditions that the changes in $x_i$ and $x_j$ do not result in any change in utility (that is, movement along a given indifference curve is involved), and only $x_i$ and $x_j$ change, while all other commodities remain constant.

The usual definition of this concept for two commodities seems restrictive given that commodity spaces can be larger. Another way to think about this could be in terms of two commodity bundles. We define the generalized marginal rate of substitution (GMRS) at $x$ as the real number such that the losses for commodity bundle $w_1$ can be compensated with this amount of another given commodity bundle $w_2$ (keeping utility constant).

This notion of compensation has precedence in the properness property introduced by Mas-Colell (1986). The GMRS and the property of uniform properness are related for a preference relation that can be represented by a differentiable utility function. Precisely, if $\succeq$ is a preference relation defined on an open subset $X$ of a Banach space $E$, then $v$ is an uniform proper vector for $\succeq$ iff $\mathrm{GMRS}_{v,w}(x)$ is bounded from below by a positive number. This will be true for all $x \in X$ and $w \in E_+$, $\|w\| = 1$.

It is observed that the boundedness of the marginal rate of substitution is necessary for the existence of equilibrium in economies with infinitely many commodities (as is noted by Peleg and Yaari, 1970, Bewley, 1972 with the adequate assumptions, Mas-Colell, 1986 with the properness property, and Rustichini and Yannelis, 1991 among others).

## 2. The generalized marginal rate of substitution

In this section we consider the commodity space $E = \mathbb{R}^n$. This is later generalized to Banach lattices.

Let $X \subset \mathbb{R}^n_+$ be an open and convex set, $x_o \in X$ and let $u: X \to \mathbb{R}^n$ be a differentiable utility function which represents a continuous, monotone and strictly convex preference relation $\succeq$ defined on $X$. The properties of monotonicity and convexity guarantee the strict positivity of the partial derivatives of the utility function. This is essential to define the GMRS.

Consider two linearly independent vectors $w_1, w_2 \in \mathbb{R}^n_+$, where $\|w_1\| = \|w_2\| = 1$, and complete $\{w_1, w_2\}$ to a positive basis $B = \{w_1, \dots, w_n\}$ for $\mathbb{R}^n$. Let $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ be the linear function defined by $\varphi(e_i) = w_i$, where $\{e_1, \dots, e_n\}$ is the standard basis for $\mathbb{R}^n$. Thus, $\varphi$ is a $C^\infty$ diffeomorphism and its associated matrix is the matrix of change from $B$ to the standard basis. Let the function $F = u \bigcirc \varphi$, thus

$$\frac{\partial F}{\partial y_i}\left(\varphi^{-1}(x_o)\right) = \sum_{j=1}^{n} w_i^j \frac{\partial u}{\partial x_j}(x_o) \text{ with } w_i = \left(w_i^1, \cdots, w_i^n\right).$$

Given that $(\partial F)/(\partial y_i)(\varphi^{-1} x_o) > 0$, we get the implicit function $y_2 = g(y_1, y_3, \dots, y_n)$ verifying

$$\frac{\partial g}{\partial y_1}\left(\varphi^{-1}(x_o)\right) = -\frac{\dfrac{\partial F}{\partial y_1}\left(\varphi^{-1}(x_o)\right)}{\dfrac{\partial F}{\partial y_2}\left(\varphi^{-1}(x_o)\right)} = -\frac{\sum_{j=1}^{n} w_1^j \dfrac{\partial u}{\partial x_j}(x_o)}{\sum_{j=1}^{n} w_2^j \dfrac{\partial u}{\partial x_j}(x_o)} \equiv \frac{\partial y_2}{\partial y_1}. \qquad (1)$$

We define the GMRS between the commodity bundles $w_1$ and $w_2$ at $x_o$ as the real number $-(\partial y_2/\partial y_1)$. This is the amount of the commodity bundle $w_2$ that would be necessary to compensate for the loss of the commodity bundle $w_1$, keeping $F$ constant and equal to $F(\varphi^{-1}(x_o))$. This quotient, which depends only on the bundles $w_1$, $w_2$ and $x_o$, and not on the chosen utility function, is denoted by $\text{GMRS}_{w_1,w_2}(x_o)$. This is, in fact, a generalization of the marginal rate of substitution between two commodities. Note that if $w_1 = e_i$ and $w_2 = e_j$, this concept is in accordance with the definition of $\text{MRS}_{i,j}(x_o)$.

Consider some smooth utility function $u$ representing the preference relation $\succeq$. Now, we check that $\text{GMRS}_{w_1,w_2}(x_o)$ can be defined as the quotient of the directional derivatives of $u$ in the directions of $w_1$ and $w_2$, respectively. From Eq. (1) we obtain

$$\text{GMRS}_{w_1,w_2}(x_o) = \frac{w_1^1 \dfrac{\partial u}{\partial x_1}(x_o) + \cdots + w_1^n \dfrac{\partial u}{\partial x_n}(x_o)}{w_2^1 \dfrac{\partial u}{\partial x_1}(x_o) + \cdots + w_2^n \dfrac{\partial u}{\partial x_n}(x_o)} = \frac{D_{w_1} u(x_o)}{D_{w_2} u(x_o)}.$$

The economic interpretation of the GMRS can now be obtained by using the usual interpretation of the MRS. Dividing the earlier expression by $(\partial u/\partial x_1)(x_o)$, we obtain

$$\text{GMRS}_{w_1,w_2}(x_o) = \frac{w_1^1 + w_1^2 \text{MRS}_{2,1}(x_o) + \cdots + w_1^n \text{MRS}_{n,1}(x_o)}{w_2^1 + w_2^2 \text{MRS}_{2,1}(x_o) + \cdots + w_2^n \text{MRS}_{n,1}(x_o)}.$$

The interpretation of this is as follows. To compensate for the loss of a unit of the second commodity (keeping the utility constant), a transfer of $\text{MRS}_{2,1}(x_o)$ units of the first commodity is necessary. Thus, for the loss of $w_1^2$ units of the second commodity, $w_1^2 \text{MRS}_{2,1}(x_o)$ units of the first commodity are necessary, and so on. In general, for the loss of a commodity bundle $w_1$ a transfer of $A = w_1^1 + w_1^2 \text{MRS}_{2,1}(x_o) + \dots + w_1^n \text{MRS}_{n,1}(x_o)$ units of the first commodity are necessary.

In the same way, we obtain that a commodity bundle $w_2$ could be compensated with the transfer of $B = w_2^1 + w_2^2 \text{MRS}_{2,1}(x_o) + \dots + w_2^n \text{MRS}_{n,i}(x_o)$ units of the first commodity, i.e., each unit of the first commodity is equivalent to $1/B$ times the commodity bundle $w_2$. Thus, $\text{GMRS}_{w_1,w_2}(x_o) = A/B$ is the amount of the commodity bundle $w_2$ that is necessary to compensate the loss of the commodity bundle $w_1$.

In the general case the commodity space $E$ is a Banach lattice, given two *linearly independent* vectors $w_1,w_2$, consider $E - \langle w_1 \rangle \times \langle w_2 \rangle \times H$ where $H$ is the topological complement of $\langle w_1 \rangle \times \langle w_2 \rangle$ in $E$. Each vector $x \in E$ has a unique representation $x = (x^1,x^2,x^3) \in \mathbb{R} \times \mathbb{R} \times H$. As before, consider a differentiable utility function $u$ defined on an open set $X \subset E = \langle w_1 \rangle \times \langle w_2 \rangle \times H$. Then, $(\partial u / \partial x^1)(x_o) = D_{w_1} u(x_o)$ and $(\partial u / \partial x^2)(x_o) = D_{w_2} u(x_o)$. By the implicit function theorem there exist open neighbourhoods $V$ of $x_o$ in $E$ and $W$ of $(x_o^1,x_o^3)$ in $\langle w_1 \rangle \times H$, respectively, and a differentiable function $g: W \to \mathbb{R}$ such that

$$(x^1,x^2,x^3) \in V; u(x^1,x^2,x^3) = u(x_o) \Leftrightarrow (x^1,x^3) \in W; g(x^1,x^3) = x^2$$

and

$$\frac{\partial x^2}{\partial x^1}(x_o) \equiv D_1 g(x_o^1,x_o^3) = -\frac{\dfrac{\partial u}{\partial x^1}(x_o)}{\dfrac{\partial u}{\partial x^2}(x_o)}$$

Thus, for a continuous, monotone and strictly convex preference relation defined on an open and convex subset $X$ of a Banach lattice $E$ we have the following.

**Definition 1.** The GMRS between two commodity bundles $w_1,w_2 \in E$, at a point $x \in X$ relative to the preference relation $\succeq$ is defined by the quotient $\mathrm{GMRS}_{w_1,w_2}(x) = (D_{w_1} u(x))/(D_{w_2} u(x)) = (Du(x)(w_1))/(Du(x)(w_2))$, where $u$ is any differentiable utility function representing $\succeq$.

Note that, as was said before, in the case where $E = \mathbb{R}^n$, if $w_1 = e_i$ and $w_2 = e_j$, $\mathrm{GMRS}_{w_1,w_2}(x)$ coincides with the usual $\mathrm{MRS}_{ij}(x)$ definition.

Finally, it is necessary to prove that the definition of the GMRS does not depend on the differentiable utility function $u$ which represents $\succeq$. This is a consequence of the next lemma.

Let $u$ and $v$ be two differentiable utility functions representing the preference relation $\succeq$.

**Lemma 1.** *If $Du(x) \neq 0$, $Dv(x) \neq 0$, for all $x \in X$; then there exists an increasing differentiable function $\varphi : v(X)\ \mathbb{R}$, such that $\varphi'(t) \neq 0$, for all $t \in v(X)$, and $u = \varphi \bigcirc v$.*

*Proof.* Let $t \in v(X)$. Consider $x \in X$ such that $t = v(x)$ and define $\varphi(t) = u(x)$. Then $\varphi$ is well defined and also is the only function verifying $u = \varphi \bigcirc v$. We will prove that this function is differentiable, increasing and $\varphi'(t) \neq 0$, for all $t \in v(X)$.

Take $t \in v(X)$ and $x \in X$ such that $t = v(x)$. Let $w$ be a unitary vector such that $Dv(x)(w) \neq 0$. Let $q = Dv(x)(w) \in \mathbb{R}$ and $F: E_o \times \mathbb{R} \to \mathbb{R}$ defined by $F(\lambda,s) = v(x + \lambda q w) - s$; where $E_o$ is a suitable neighbourhood of 0. Observe that $D_\lambda F(0,t) = q^2 \neq 0$, and then, by the implicit function theorem, there exist an open neighbourhood $U$ of $t$ in $\mathbb{R}$ and a differentiable function, $\lambda: U \to \mathbb{R}$ verifying $\lambda(t) = 0$ and $v(x + \lambda(s)q) - s = 0$ for in all $s \in U$. Moreover, $\lambda(t) = -1/q_2$.

Now, defining $\mu: U \to \mathbb{R} \cdot \mu(s) = u(x + \lambda(s)q)$. Thus, $u(x + \lambda(s)q) = \mu(v(x + \lambda(s)q))$, for all $s \in U$. Consequently, $\mu$ coincides with $\varphi$ in a neighbourhood of $t$; and, because of $\mu$ is differentiable in $U\mu'(t) = (1/q^2)Dv(x)(w)Du(x)(w) > 0$, $\varphi$ verifies the theorem.

Then, by the lemma,

$$\frac{Du(x_o)(w_1)}{Du(x_o)(w_2)} = \frac{D\varphi(v(x_o)) \bigcirc Dv(x_o)(w_1)}{D\varphi(v(x_o)) \bigcirc Dv(x_o)(w_2)} = \frac{\varphi'(v(x_o))Dv(x_o)w_1)}{\varphi'(v(x_o))Dv(x_o)w_2)}$$
$$= \frac{Dv(x_o)(w_1)}{Dv(x_o)(w_2)}.$$

## 3. GMRS and the properness property

In this section we relate GMRS and the properness property. The properness property is quite close to the idea that the GMRS is bounded from below. In fact, Mas-Colell (1986) introduced this property to impose a priori bounds on the MRS.

Following Mas-Colell (1986), a preference relation $\succeq$, defined on a consumption set $X$, is proper at $x \in X$ whenever there exists some $v > 0$ and some neighbourhood of zero $V$ such that $x - \alpha v + z \succeq x$ with $\alpha > 0$ implies $z \notin \alpha V$. This property expresses the economic intuition that the commodity bundle $v$ is desirable, in the sense that the loss of an amount $\alpha v$ cannot be compensated with an additional amount $\alpha z$ of any commodity bundle $z$ which is sufficiently small.

Let $w$,$v$ be two linearly independent commodity bundles in $E$. Then the economic interpretation says that $\mathrm{GMRS}_{v,w}(x_o)w$ compensates the loss of a unit of the vector $v$. Thus, if $v$ is a proper vector at $x_o$, and $V = B(0,\varepsilon)$ is the open neighbourhood associated, then $\mathrm{GMRS}_{v,w}(x_o)w \notin V$. Supposing $w$ is a unitary vector, we have $\mathrm{GMRS}_{v,w}(x_o) \succeq \varepsilon$.

We show this with the help of the following example. The idea is formally stated in Theorem 1.

**Example 1.** On $X = \mathbb{R}^2_{++}$ the preference relation represented by $u(x,y) = xy$ is, obviously, proper at each bundle $x \in X$. Moreover $v = \sqrt{2}/2, \sqrt{2}/2$ is a proper vector and $\mathrm{GMRS}_{v,w}(x) \succeq \sqrt{2}/2$ for all $x \in X$ and $w \in \mathbb{R}^n_+$, $\|w\| = 1$. In fact, the biggest $\varepsilon > 0$ such that $B((0,0),\varepsilon)$ is the open set corresponding to $v$ is $\inf\{\mathrm{GMRS}_{v,w}(x): x \in X,\ w \in E_+,\ \|w\| = 1\} = \sqrt{2}/2$. This can be easily checked.

Next, we prove that this result is always true. Before we do so, it should be noted that a preference relation $\succeq$ defined on a subset $X$ of a Banach lattice $E$, is proper at $x$ if and only if there exist some positive vector $v > 0$, and some real number $\varepsilon > 0$, such that $x - \alpha v + z \succeq x, \alpha > 0$ and $z \in E$ implies $\|z\| \geq \alpha\varepsilon$. Any vector $v$ that satisfies this property will be referred to as a proper vector for $\succeq$ at $x$.

**Theorem 1.** *Let E be a Banach lattice. Let X be an open subset of E and* $u: X \to \mathbb{R}$ *be a differentiable utility function representing a monotone and convex preference-relation* $\succeq$ *such that* $Du(x_o) > 0$.

Then $v \in E_+$ is a proper vector for $\succeq$ at $x_o \in X$ iff there exists $\varepsilon > 0$ such that $\mathrm{GMRS}_{v,w}(x_o) \succeq \varepsilon$, for all $w \in E_+$, $\|w\| = 1$.

*Proof*. Suppose that $v \in E_+$ is a proper vector at $x_o$ with $B(0,\varepsilon)$ as the associated open set. Let $w \in E_+$, $\|w\| = 1$.

It is known that $\mathrm{GMRS}_{v,m}(x_o) = -D_1\, g(x_o^2, x_o^3) = \lim_{h \to 0}(g(x_o^2 + h,\ x_o^3) - g(x_o^2,\ x_o^3))/(-h)$, where $g$ is the implicit function defined by $u: X \subset \langle w \rangle \times \langle v \rangle \times H \equiv E \to \mathbb{R}$ in some open neighbourhood of $x_o = x_o^1 w + x_o^2 v + x_o^3 \equiv (x_o^1, x_o^2, x_o^3) \in \langle w \rangle \times \langle v \rangle \times H$.

Moreover, for $h$ small enough, it is verified that $(x_o^2 + h, x_o^3)$ belongs to the domain of $g$ and, consequently, $g(x_o^2 + h, x_o^3)w + (x_o^2 + h)v + x_o^3 \sim x_o$, then

$$x_o \sim x_o^1 w - g\left(x_o^2, x_o^3\right)w + g\left(x_o^2 + h, x_o^3\right)w + x_o^2 v + hv + x_o^3$$
$$= x_o + \left(g\left(x_o^2 + h, x_o^3\right) - g\left(x_o^2, x_o^3\right)\right)w + hv.$$

For $h < 0$, the properness property guarantees that $|g(x_o^2 + h, x_o^3) - g(x_o^2, x_o^3)| \geq -h\varepsilon = |h|\varepsilon$. Thus, for $h < 0$ in a neighbourhood of zero, $|(g(x_o^2 + h, x_o^3) - g(x_o^2, x_o^3))/(h)| \geq \varepsilon$ and consequently,

$$|\mathrm{GMRS}_{v,w}(x_o)| = |D_1 g\left(x_o^2, x_o^3\right)| \geq \varepsilon.$$

On the other hand, for $h \neq 0$, sufficiently small, it is verified that $(g(x_o^2 + h, x_o^3) - g(x_o^2,\ x_o^3))/(h) < 0$. This follows because if $h < 0$ and $g(x_o^2 + h, x_o^3) < g(x_o^2, x_o^3)$, as $w, v \in E_+$, the monotonicity property states that $(x_o^1, x_o^2, x_o^3) = (g(x_o^2, x_o^3), x_o^2, x_o^3) \succ (g(x_o^2 + h, x_o^3), x_o^2 + h, x_o^3)$, which is impossible. Analogous for $h > 0$. Thus, $\mathrm{GMRS}_{v,w}(x_o) \geq 0$, and $\mathrm{GMRS}_{v,w}(x_o) \succeq \varepsilon$ for all unitary vector $w \in E_+$.

Suppose now that there exist $\varepsilon > 0$ and $v \in E_+$ such that $\mathrm{GMRS}_{v,w}(x) \succeq \varepsilon$, for all $x \in X$ and for all $w \in E_+$, $\|w\| = 1$. We prove that $v$ is a proper vector at $x_o$.

Let $z \in E_+$, $\|z\| < \varepsilon$. First we prove $x_o > x_o + \alpha(z - v)$, for all $\alpha > 0$ such that $x_o + \alpha(z - v) \in X$.

Consider $z' = (1/\|x\|)z$, we then know that $\mathrm{GMRS}_{v,z'}(x_o) = D_1 g(x_o^2, x_o^3)$ where $x_o = x_o^1 z' + x_o^2 v + x_o^3 \in \langle z' \rangle \times \langle v \rangle \times H \equiv E$ and $g$ is the implicit function defined on an open neighbourhood of $(x_o^2, x_o^3)$ by $g(x_2, x_3)z' + x_2 v + x_3 \sim x_o$ (that is $u(g(x_2, x_3), x_2, x_3) = u(x_o)$).

As $\lim_{h\to 0}(g(x_o^2+h,x_o^3)-g(x_o^2,x_o^3))/(-h)\geq \varepsilon$, there exists $r>0$ such that if $h\in(-r,r)$, then $(g(x_o^2+h,x_o^3)-g(x_o^2,x_o^3))/(-h)\geq\varepsilon$. Consider $h\in(-r,0)$, then

$$x_o \sim g\left(x_o^2+h,x_o^3\right)z'+\left(x_o^2+h\right)v+x_o^3=g\left(x_o^2+h,x_o^3\right)z'-g\left(x_o^2,x_o^3\right)z' + g\left(x_o^2,x_o^3\right)z'+x_o^2v+hv+x_o^3=x_o-h\left(\frac{g\left(x_o^2+h,x_o^3\right)-g\left(x_o^2,x_o^3\right)}{-h}z' - v\right)\geq x_o-h(\varepsilon z'-v)>x_o-h\|z\|z'-v)=x_o-h(z-v).$$

Consider and $\alpha_0=r/2$, thus we have proved that for each $z\in E_+$ with $\|z\|<\varepsilon$, there exists and $\alpha_0>0$ such that $x_o \succ x_o+\alpha(z-v)$, for all $\alpha\in$ .

On the other hand, for $\alpha>\alpha_0$ such that $x_o+\alpha(z-v)\in X$,it is verified that $x_o\succ x_o+\alpha(z-v)$, as a consequence of the convexity of the preference relation.

Now, let $z\in E$. Consider $z^+=\sup\{z,0\}\in E_+$. Then, there exists $\alpha_0>0$ such that $x_o\succ x_o+\alpha(z^+-v)$, for all $\alpha\in(0,\alpha_0)$. Thus, we have proved $x_o\succ x_o+\alpha(z^+-v)$, and, by the monotonicity, $x_o\succ x_o+\alpha(z-v)$.

By using again the convexity, we obtain $x_o\succ x_o+\alpha(z-v)$ for all $\alpha>0$ such that $x_o+\alpha(z-v)\in X$. That is, if $\alpha>0$ and $x_o-\alpha v+\alpha z\succeq x_o$ then $\|z\|\geq\varepsilon$.

Note that a preference relation $\succeq$ defined on the consumption set $X$, is uniformly proper whenever there exist some $v>0$ and some neighbourhood $V$ of zero such that $x-\alpha v+z\succeq x$ with $\alpha>0$ and $x\in X$, implies $z\notin\alpha V$. Thus, as a corollary of the theorem above, we obtain a necessary and sufficient condition for the uniform properness property

**Corollary 1**. *Let E be a Banach lattice. Let X be an open subset of E and u:X → ℝ be a differentiable utility function representing a monotone and convex preference relation $\succeq$ such that Du(x) > 0, for all $x\in X$.*

Then $v\in E_+$ is a uniform proper vector iff there exists $\varepsilon>0$ such that $\mathrm{GMRS}_{v,w}(x)\geq\varepsilon$, for all $x\in X$, $w\in E_+$, $\|w\|=1$.

Two immediate consequences of this corollary are as follows: (i) if $v$ is a vector, such that there exist $x\in X$ and a unitary vector $w\in E_+$ with $\mathrm{GMRS}_{v,w}(x)=0$, then $v$ is not a uniform proper vector; (ii) if $v$ is a uniform proper vector, the biggest $\varepsilon>0$ such that $B(0,\varepsilon)$ is the corresponding open set is $\inf\{\mathrm{GMRS}_{v,m}(x): x\in X,\ w\in E_+,\ \|w\|=1\}$.

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