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Weakly smooth preferences on Banach lattices

M. Besada, C. Váquez*

Facultade de Económicas, Universidade de Vigo, Aptdo. 874, 36200 Vigo, Spain

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Abstract

Given a preference relation defined on a subset of a Banach lattice, verifying the usual properties and an additional assumption of no-discrimination, we construct a smooth function that can be used instead of the utility function in many cases. © 1998 Elsevier Science S.A.

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1. Introduction

The existence of a differentiable utility function representing a preference relation is useful on consumer behaviour and on equilibria theory. In fact, this property can be needed for linear approaching problems, for the calculus of marginal rate of substitution, for the first or second-order conditions to maximize the consumer's utility subject to constraints, etc.

The continuous representation theorems of Eilemberg (1941); Debreu (1959) do not provide sufficient conditions for the differentiability of the utility function. The first answer to this question, for the finite dimensional case, was given by Debreu (1972), (1976), who examines conditions on preferences which allow them to be represented by C^r utility functions. In the commodity space $\mathbb{R}^n$ he shows that a preference relation can be represented by a C^r utility function without critical points if, and only if, the preference relation is continuous, monotone and the set $I=\{(x,y):x\sim y\}$ is a C^r submanifold in $\mathbb{R}^{\{2n\}}$. The above condition requires not only that individual indifference sets be smooth, but that they vary smoothly. Neilson (1991) gives a weaker notion of smoothness for preference orderings that comes from simply dropping the requirement that the indifference sets vary smoothly.

This paper pursues sufficient conditions on the preference relation to ensure the existence of such kind of function (the surrogate function) for the case of an infinite dimensional commodity space. In order to do it we impose the property which we have named no-discrimination for the commodities. The surrogate function can be used to obtain the economic properties of the indifference surface, for instance the marginal rate of substitution.

*Corresponding author. Tel.: (34-86) 812450; fax: (34-86) 812401; e-mail: cvazquez@uvigo.es

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Now let us comment on the difficulties of generalizations on the properties of the finite dimensional economies which are based on differentiability to infinite dimensional economies. We would like to remark that in order to talk about differentiability concepts we need the consumption set to be an open set. Typically the preference relation is defined on a subset of the positive orthant of the commodity space. This is rarely true in infinite dimensional spaces in which the positive orthant is typically (except for ℓ_∞) of empty interior. We fit our assumption to the general case; we restrict ourselves to Banach lattices, thus there are two alternative kind of spaces depending on whether the positive orthant has empty interior or not. In the first case, the preference relation will be defined on an open neighbourhood of the positive cone and in the second case, the consumption set is the interior of the positive cone. This assumption has been used, for instance, in Araujo (1988); Besada and Vazquez (in press). In this way, in Richard and Zame (1986) the extension of uniformly proper preferences to a certain larger set with non-empty interior is developed.

2. Definitions

A preference relation defined on the set X is a binary relation defined on X, say $\succeq$, which is reflexive, transitive and complete. A utility representation of the preference relation $\succeq$ is a function $u{:}X \to \mathbb{R}$ such that $x \succeq y$ if, and only if, $u(x) \geq u(y)$.

The usual assumptions about preference relations that are used in our work are continuity, monotonicity and strictly convexity. A preference relation $\succeq$ defined on X is said to be continuous if for each $x \in X$ the sets $\{z \in X{:}z \succeq x\}$ and $\{z \in X{:}x \succeq z\}$ are closed in X. A preference relation $\succeq$ defined on X is said to be monotone whenever x, $y \in X$ and $x > y$ imply $x \succ y$. A preference relation defined on a convex set $X \subset E$ is said to be strictly convex whenever $y \succeq x$ and $z \succeq x$ in X and $0 < \alpha < 1$ imply $\alpha y + (1-\alpha)z \succ x$.

We generalized Neilson's notion of weakly smooth preferences to infinite dimensional commodity spaces. A preference relation defined on an open set $X \subset E$ is said to be weakly C^r if all its indifference sets are one-codimensional C^r submanifolds on E. That is, if for each $x \in X$ the set $I_x = \{z \in X{:}z \sim x\}$ is a one-codimensional C^r submanifold on E.

Let $\succeq$ be a weakly C^r preference relation on X and $x \in X$ be a commodity bundle. We construct a C^r function $v{:}X \to \mathbb{R}$ having the indifference set containing x, I_x, as a level set. This function is not a utility function, but can be used instead of it for all properties relative to the indifference set I_x.

Definition 1. Let $\succeq$ be a weakly C^r preference relation defined on an open subset X of a Banach lattice E. A C^r homogeneous function of degree one $v{:}X \to \mathbb{R}$ having the indifference set I as a level set is called a C^r surrogate function associated to the indifference set I for the preference relation $\succeq$.

The existence of a surrogate function associated to a previously fixed indifference set I is proved, under some assumptions, in Theorem 1. We will use the following property.

Definition 2. A preference relation $\succeq$ on $X \subset E$ is said to be no-discriminatory if for each $x,y \in X$, there exist $r,s \in \mathbb{R}$ such that $rs > 0$ and $sy \succ x \succ ry$.

The following examples illustrate this concept and its relation with other properties.

Example 1.

1. Any monotone preference relation $\succeq$ defined on $\mathbb{R}_{++}{}^{n}$ is no-discriminatory.
2. Any continuous and monotone preference relation $\succeq$ defined on $\ell_p{}^{++} = \{x = (x_k) \in \ell_p : x_k > 0$ for all $k\}$, $1 \leq p < \infty$, is no-discriminatory.
3. On ℓ_2^+-{0}, the preference relation given by $u(x) = (x_1)/(x_1 + 1) + \Sigma_{n=2}^{\infty} x_n^2$ is not no-discriminatory.
4. Any monotone preference relation defined on $Int(\ell_\infty^+)$ is no-discriminatory.

3. Results

In order to prove the main result of this paper we will use a property of the homogeneous functions of degree one, that we establish in Proposition 1 below. This property means that the graph of a continuous, homogeneous function of degree one which level sets are one-codimensional C^r manifolds, varies smoothly. Note that this property is very easy to prove when $E = \mathbb{R}^n$, but, for the infinite dimensional case, the proof became more difficult because of the lack of equivalence between all infinite dimensional real spaces.

Proposition 1. *Let E be a Banach lattice and $X \subset E$ be an open set. Let $f : X \to (0, +\infty)$ be a continuous, homogeneous of degree one function, and assume that all of its level sets are one-codimensional C^r submanifolds on E. Then f is C^r in X.*

Proof. Given $x \in X$. Let I_x be the level set containing x, i.e.

$$I_x = f^{-1}(k), \quad k = f(x)$$

By hypothesis, I_x is a C^r submanifold on E such that codim $(I_x) = 1$. Thus, there exist U_x neighbourhood of x in E, a closed vector subspace H on E of codimension one, and a C^r diffeomorphism $g_x : U_x \to E$ such that $g_x(U_x \cap I_x) = g_x(U_x) \cap H$. Without loss of generality, we can assume that $U_x \cap I_x \cap F = \emptyset$.

Let $L_x : E \to \mathbb{R}$ be a linear and continuous function, such that $H = L_x{}^{-1}(0)$. Moreover L_x is C^∞.

We will prove, using L_x, that it is possible to rewrite the function f on some neighbourhood of x.

Consider $s > 0$ such that $B(x,s) \subset U_x$. Define $\psi : E \times \mathbb{R} \to E$, $\psi(y,t) = ty$. It is straightforward to see that we can choose $\varepsilon > 0$ and $r > 0$ such that for $V_x = \psi((B(x,r) \cap I_x) \times (1-\varepsilon, 1+\varepsilon))$, $V_x \cap X$ is a neighbourhood of x in X.

Now the function ψ on $((B(x,r) \cap I_x) \times (1-\varepsilon, 1+\varepsilon)) \cap \psi^{-1}(X)$ is a C^r diffeomorphism over its image. This comprises a change of coordinates on $V_x \cap X$. In fact, let be $z \in V_x \cap X$, consider $(y,t) \in (B(x,r) \cap I_x) \times (1-\varepsilon, 1+\varepsilon)$ such that $x = ty$, then we have,

$$f(z) = f(ty) = tf(y) = tf(x) = tG_x(g_x(y)) = t(L_x(g_x(y)) + k)$$

$$= L_x(tg_x(y)) + tk.$$

Thus f is C^r on a neighbourhood of x. □

Corollary 1. *Let E be a Banach lattice and $X \subset E$ be an open set. Let $f{:}X \to \mathbb{R}$ be a continuous function homogeneous of degree one and such that $f(x)=0$ for all $x \in X$. If all the level sets of f are one-codimensional C^r submanifolds on E, then f is C^r in X.*

Proof. Consider the open sets $X_+=\{x \in X{:}f(x)>0\}$ and $X_-=\{x \in X{:}f(x)<0\}$. Define $f_+{:}X_+ \to (0,+\infty)$, $f_+(x)=f(x)$ and $f_-{:}X_- \to (0, +\infty)$, $f_-(x)=-f(x)$. Then f is C^r on X because it coincides with f_+ in X_+ and with $-f_-$ in X_-, which are C^r by Proposition 1. □

Theorem 1. *Let E be a Banach lattice and $\succeq$ be a continuous, strictly convex, no-discriminatory and weakly C^r preference relation defined on an open and convex subset $X \subset E$.*

Then for each indifference set I, there exists a C^r surrogate function $v{:}X \to \mathbb{R}$ associated to I.

Proof. Let $x \in X$ such that $I=I_x=\{z \in X{:}z \sim x\}$.

First of all, we prove that for each $y \in X$, there exists $t_o=0$ such that $t_o y \sim x$. Let $S=\{t=0{:}ty \succ x\}$. By hypothesis, there exist r, $s \in \mathbb{R}$ such that $rs>0$ and $sy \succ x \succ ry$, then $S=\emptyset$. If $s>r$, then, by the strict convexity of the preference ordering, S is bounded from below by r. So we can take $t_o=\inf S$. On the other hand, if $r>s$, then, by the strict convexity, S is bounded from above by r, and we take $t_o=\sup S$. In any case, we have $t_o y \sim x$.

Now we see that t_o is unique. Otherwise suppose $t_o>t_o'$ verifying $t_o' y \sim t_o y \sim x$. Let z be a $\succeq$-maximal on the set $L[t_o y.t_o']=\{sy{:}s \in [t_o', t_o]\}$, thus $z \succ ty$ for all $t \in [t_o', t_o]$. By the convexity of the preference relation, $t_o y \succ sy$ for all $s \in (-\infty, t_o') \cup (t_o,+\infty)$, and then $z \succ t_o y \succ sy$. Thus $z \succ sy$, for all $s=0$, which is a contradiction with the no-discriminatory property.

Define $v(y)=1/(t_o)$ and $w(y)=t_o y$. Then v is well defined and we have

$$v(y)=0\,,\ \text{and}\ w(y) \in I_x\,,\ \ y=v(y)w(y) \tag{1}$$

The function $v{:}X \to \mathbb{R}$ verifies:

1. v is a homogeneous degree one function, because of the uniqueness of $v(y)$ and $w(y)$ verifying Eq. (1).
2. v is continuous. In fact we prove that $t{:}X \to \mathbb{R}$, verifying $t(y)y \sim x$ is continuous. Suppose $t(y)=\inf\{t=0{:}ty \succ x\}$. Let $\varepsilon>0$, $y \in X$ and $\{y_n\}$ a sequence in X such that $\lim_{n\to\infty} y_n = y$. Take $t=0$ such that $t(y)<t<t(y)+\varepsilon$ and $ty \succ x$. By the continuity of the preference relation, there exists $n_1 \in \mathbb{N}$ such that, $ty_n \succ x$ for all $n \geq n_1$. Thus $t(y_n) \leq t < t(y)+\varepsilon$. In the same way, there exists $n_2 \in \mathbb{N}$ such that $t(y)-\varepsilon<t(y_n)$ for all $n \geq n_2$. Then $\{t(y_n)\} \to t(y)$.
3. I_x is a level set of v, because $I_x=v^{-1}(1)$. And its level sets are $tI_x=\{tz{:}z \in I_x\}$, $t=0$, that are one-codimensional C^r manifolds.

Then, by Corollary 1, $v \in C^r(X)$. □

Observe that the surrogate function v can be different for each level set. Moreover in the next proposition we set that the surrogate function associated to a level set is unique except for strict positive homothetic maps.

Proposition 2. *Let E be a Banach lattice and* $\succeq$ *be a continuous, strictly convex, no-discriminatory and weakly* C^r *preference relation defined on a convex and open subset* $X \subset E$. *Then two surrogate functions, associated to the same indifference set, I, are multiple.*

Proof. Let $v{:}X \to \mathbb{R}$ be the surrogate function associated to I as in the proof of Theorem 1; and let $\tilde{v}{:}X \to \mathbb{R}$ be another surrogate function associated to I. We prove that there exists a constant $k \in \mathbb{R}$, $k=0$, such that $\tilde{v}(z)=kv(z)$, for all $z \in X$.

Consider $x_o \in X$ such that $I_{x_o}=I$, and let $z \in X$, then

$$\tilde{v}(z) = \tilde{v}(v(z)w(z)) = v(z)\tilde{v}(w(z)) = v(z)\tilde{v}(x_o)$$

Thus, we have $k=\tilde{v}(x_o)$. □

Obviously, given a surrogate function v, associated to I, for each real number $k=0$, the function $\tilde{v}{:}X \to \mathbb{R}$, $\tilde{v}(z)=kv(z)$ is another surrogate function associated to I. So we have a characterization of the surrogate functions associated to the same indifference set.

Observe that if we suppose $I \cap E_+ = \emptyset$, the surrogate function associated to I constructed in the proof of Theorem 1, is increasing in $X \cap E_+$. Moreover, if $\tilde{v}{:}X \to \mathbb{R}$ is another surrogate function associated to I increasing in the ordering in E_+, the real number k such that $\tilde{v}=kv$ is strictly positive.

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